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Finite element modelling of wave loading on a multi-layered gassy soil seabed using complex number theory

Modélisation par éléments finis de la charge des vagues sur un fond marin à sol multicouche à l'aide de la théorie des nombres complexes

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ABSTRACT: Ocean wave loading on a gassy soil seabed, results in harmonic changes in stresses, displacements and pore water pressures over depth. For a multi-layered seabed soil with homogeneous properties within each layer, the behaviour of the dependent variables of displacement and pore pressure can be represented by a harmonic function in both horizontal space (x) and time (t). The amplitude of the horizontal & vertical displacement and pore-water pressure will vary over depth, dependent on the particular distribution of the multi-layered properties, together with the wave loading amplitude, wave number and angular velocity, together with the boundary conditions at the seabed, i.e. permeable or impermeable. To simulate the behaviour of the seabed in harmonic wave conditions, this paper presents a series of governing equations based on Biot theory for poro-elastic soil. These equations have been modified to represent the harmonic behaviour using real & imaginary complex number theory. The Finite Element Galerkin formulation combined with Green's Theorem is applied to these harmonic equations which result in six degrees of freedom per node, i.e. x-displacement, z-displacement and pore-water pressure, in both real (in-phase) & imaginary (out-of-phase) conditions. The harmonic finite element model can be used to match measured pore water pressure amplitude decay at various depths within a seabed, when subject to wave & tide loading.

RÉSUMÉ: La charge des vagues océaniques sur un fond marin gazeux entraîne des changements harmoniques dans les contraintes, les déplacements et les pressions de l'eau interstitielle en profondeur. Pour un sol de fond marin multicouche avec des propriétés homogènes au sein de chaque couche, le comportement des variables dépendantes du déplacement et de la pression interstitielle peut être représenté par une fonction harmonique à la fois dans l'espace horizontal (x) et dans le temps (t). L'amplitude du déplacement horizontal et vertical et de la pression de l'eau interstitielle variera en fonction de la profondeur, en fonction de la distribution particulière des propriétés multicouches, ainsi que de l'amplitude de charge des vagues, du nombre de vagues et de la vitesse angulaire, ainsi que des conditions aux limites du fond marin, c'est-à-dire perméable ou imperméable. Pour simuler le comportement des fonds marins dans des conditions de vagues harmoniques, cet article présente une série d'équations régulatrices basées sur la théorie de Biot pour les sols poro-élastiques. Ces équations ont été modifiées pour représenter le comportement harmonique à l'aide de la théorie des nombres complexes réels et imaginaires. La formulation de Galerkin par éléments finis combinée au théorème de Green est appliquée à ces équations harmoniques qui se traduisent par six degrés de liberté par nœud, c'est-à-dire le déplacement x, le déplacement z et la pression de l'eau interstitielle, dans des conditions réelles (en phase) et imaginaires (hors phase). Le modèle d'éléments finis harmoniques peut être utilisé pour faire correspondre la décroissance mesurée de l'amplitude de la pression de l'eau interstitielle à différentes profondeurs dans un fond marin, lorsqu'il est soumis à une charge de vagues et de marée.

Keywords: Wave loading; marine sediment; gassy soil; multi-layer; finite element modelling.

1 INTRODUCTION

This paper describes a Finite Element formulation of the differential equations that govern the wave-induced stresses, displacements and pore-water pressures in a multi-layered or anisotropic poro-elastic soft gassy soil seabed when subject to surface harmonic wave loading.

The harmonic stress, displacement and pore-water pressure behaviour of a gassy soil can be simulated under conditions that soil properties are homogeneous in the horizontal x and y directions, but can be variable, as in a multi-layered soil, in the vertical z-direction.

Under the condition that the total stress and water pressure loading on the seabed is harmonic in both timeand x-direction, that is a combination of a cosine and sine loading, the behaviour of each dependent variable is also harmonic in both time and the x-direction, and is governed by the wave period and wavelength of the loading wave; multiplied by the peak amplitude of the dependent variable. This is based on the equations by Yamamoto et al. (1978) and Madsen (1978).

When a combination of cosine and sine components of each dependent variable is replaced with complex number theory, this enables application of the Galerkin Finite Element formulation combined with Green's Theorem, to the equations in the vertical dimension. This requires solving six degrees of freedom per node.

2 DIFFERENTIAL EQUATIONS FOR MULTI-LAYER & ANISOTROPIC PORO-ELASTIC SOIL CONDITIONS

Based on the Biot (1941) equations that govern the displacement, stresses and pore water pressures in a poro-elastic soil, as modified by Thomas (1987) for a Double Compressibility Gassy Soil, the equations of equilibrium in the x-direction, equilibrium in the z-direction, and continuity principles incorporating Darcy's Law of pore-fluid flow through a deformable poro-elastic medium, the following differential equations can be written for small strain conditions:

$$(\lambda + 2G)\frac{\partial^2 w_x}{\partial x^2} + G\frac{\partial^2 w_x}{\partial z^2} + \lambda\frac{\partial^2 w_z}{\partial x \partial z} + G\frac{\partial^2 w_z}{\partial z \partial x} - B\frac{\partial u_w}{\partial x} = 0 \quad (1)$$

$$(\lambda + 2G)\frac{\partial^2 w_z}{\partial z^2} + G\frac{\partial^2 w_z}{\partial x^2} + \lambda\frac{\partial^2 w_x}{\partial z \partial x} + G\frac{\partial^2 w_x}{\partial x \partial z} - B\frac{\partial u_w}{\partial z} = 0 \quad (2)$$

$$\frac{\partial}{\partial x}\left[\frac{k_{xx}}{\gamma_w}\frac{\partial u_w}{\partial x} + \frac{k_{xz}}{\gamma_w}\frac{\partial u_w}{\partial z}\right] + \frac{\partial}{\partial z}\left[\frac{k_{zx}}{\gamma_w}\frac{\partial u_w}{\partial x} + \frac{k_{zz}}{\gamma_w}\frac{\partial u_w}{\partial z}\right] = C\frac{\partial u_w}{\partial t} + B\frac{\partial}{\partial t}\left[\frac{\partial w_x}{\partial x} + \frac{\partial w_z}{\partial z}\right] \quad (3)$$

where w_x and w_z are the soil displacements in the x-horizontal and the z-vertical directions respectively, t is time, u_w the pore-water pressure within the soil matrix, k_{ij} the coefficient of permeability tensor, λ the Lame's constant and G the shear modulus. The gassy soil parameters B and C are derived from gassy soil theory (Thomas, 1987) as follows:

$$B = K^o/(K^o + K") = (1 + K"/K^o)^{-1} \quad (4)$$

$$C = (K^o + K")^{-1} + (1 - H)c_w n_w \quad (5)$$

The parameter K^o is the component of undrained bulk modulus governed by compression of the discrete gas bubbles, caused by an increase in total stress. Contributing to undrained compressibility is also water compressibility, c_w, water porosity, n_w, and Henry's constant H. The parameter $K"$ is the component of drained bulk modulus governed by consolidation of the saturated soil matrix. This results in two independent strain components, (i) caused by an increase in the total stress σ^o, and (ii) caused by an increase in the operative stress (σ^o - u_w).

The modified equilibrium equations for a poro-elastic gassy soil are based on a Double Compressibility Model (Thomas, 1987) where the compression of the discrete gas bubbles is governed by the increase in the pressure of the gas bubbles, which in turn is governed by the average total stress within the soil. The compression of the surrounding saturated soil matrix, however, is governed by the difference between the total stress and the pore water pressure.

This difference between total stress and pore water pressure (σ^o - u_w) is normally known as *effective stress* (Terzaghi, 1944). However, for a gassy soil, effective stress does not, on its own, govern the total volume change. For gassy soil, this difference between the mean total stress on a soil sample and the water pressure has been coined, the *operative stress* (Sills et al, 1991). This results in two independent components of strain caused by (i) an increase in the total stress σ^o, and (ii) an increase in the operative stress (σ^o - u_w),

$$\epsilon^o = \Delta V^o/V = \Delta\sigma^o/K^o \quad (6)$$

$$\frac{\epsilon = \Delta V}{V} = \Delta\sigma"/K = (\Delta\sigma^o - \Delta u_w)/K \quad (7)$$

where ϵ^o is the volumetric strain due to the compression of the discrete gas bubbles caused by an increase in total mean stress $\Delta\sigma^o$, with $\epsilon"$ as the volumetric strain due to the compression of the soil matrix caused by an increase in operative stress $\Delta\sigma"$.

The main principle of the Double Compressibility Model (Thomas, 1987) for a gassy soil is that these two components of volumetric strain for a gassy soil, ϵ^o and, $\epsilon"$, can be added directly together to provide the total volumetric strain, as follows:

$$\epsilon = \epsilon^o + \epsilon" = \Delta\sigma^o/K^o + \Delta\sigma"/K" = \Delta\sigma^o/K^o + (\Delta\sigma^o - \Delta u_w)/K" \quad (8)$$

from which the change in total stress can be defined in terms of strain and pore pressure as:

$$\Delta\sigma^o = (\epsilon + \Delta u_w/K")/(1/K^o + 1/K") \quad (9)$$

3 HARMONIC APPROXIMATION OF GOVERNING EQUATIONS

For wave loading on the seabed when travelling in the x-direction along the surface of the seabed as depicted in Figure 1; under poro-elastic conditions, based on the equations by Yamamoto et al. (1978) and Madsen (1978), any variable ψ which is dependent on x, z and t can be defined by a variable dependent in z only, multiplied by a harmonic function dependent on x & t:

$$\psi(x,z,t) = \psi_c + i\,\psi_s \tag{10}$$

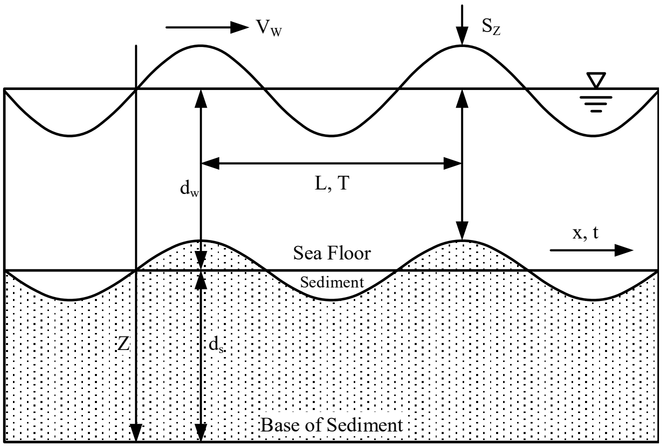

Figure. 1. Harmonic wave loading on a seabed sediment.

where:

$$\psi_c = \overline{\psi}(z)\cos(ax - \omega t) \tag{11}$$

$$\psi_s = \overline{\psi}(z)\sin(ax - \omega t) \tag{12}$$

and where $\overline{\psi}(z)$ is the peak variable function amplitude at depth z, from which:

$$\overline{\psi}(z) = \sqrt{\psi_c^2 + \psi_s^2} \tag{13}$$

$$\frac{\partial\psi}{\partial x} = +ia\psi = +ia\psi_c - a\psi_s \tag{14}$$

$$\frac{\partial\psi}{\partial t} = -i\omega\psi = -i\omega\psi_c + \omega\psi_s \tag{15}$$

where $\overline{\psi}$ is the amplitude of the harmonic variable at depth z, with a the wave number $2\pi/L$, with ω the angular velocity $2\pi/T$, and with L the wavelength and T the wave period.

This results in the harmonic displacements and pore-water pressure inhomogeneous, anisotropic soil sediment, being represented as follows:

$$w_x(x,z,t) = w_{xc} + iw_{xs} = \overline{w}_x(z)[\cos(ax - \omega t) + i\sin(ax - \omega t)] \tag{16}$$

$$w_z(x,z,t) = w_{zc} + iw_{zs} = \overline{w}_z(z)[\cos(ax - \omega t) + i\sin(ax - \omega t)] \tag{17}$$

$$u_w(x,z,t) = u_{wc} + iu_{ws} = \overline{u}_w(z)[\cos(ax - \omega t) + i\sin(ax - \omega t)] \tag{18}$$

Substituting the above three harmonic functions of x, z displacements and pore pressure into the governing equilibrium and continuity equations (1), (2) & (3), produces three differential equations, written as a mixture of real & imaginary harmonic variables:

$$-a^2(\lambda + 2G)w_x + G\frac{\partial^2 w_x}{\partial z^2} + ia(\lambda + G)\frac{\partial w_z}{\partial z} - iaBu_w = 0 \tag{19}$$

$$+(\lambda + 2G)\frac{\partial^2 w_z}{\partial z^2} - a^2 G w_z + ia(\lambda + G)\frac{\partial w_x}{\partial z} - B\frac{\partial u_w}{\partial z} = 0 \tag{20}$$

$$-a^2\frac{k_{xx}}{\gamma_w}u_w + \frac{k_{zz}}{\gamma_w}\frac{\partial^2 u_w}{\partial z^2} + ia\frac{k_{xz}}{\gamma_w}\frac{\partial u_w}{\partial z} + ia\frac{k_{zx}}{\gamma_w}\frac{\partial u_w}{\partial z} = -i\omega C u_w + a\omega B w_x - i\omega B\frac{\partial w_z}{\partial z} \tag{21}$$

4 GALERKIN FINITE ELEMENT FORMULATION

Applying the Galerkin Finite Element formulation to the above equations in the vertical dimension, combined with Green's Theorem, results in:

$$\int \left[a^2(\lambda + 2G)N_I N_J\, w_{xJ} + G\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}w_{xJ} + iaG\frac{\partial N_I}{\partial z}N_J w_{zJ} - ia\lambda\, N_I\frac{\partial N_J}{\partial z}w_{zJ} + iaB\, N_I N_J\, u_{wJ}\right]dz = S_{xI} \tag{22}$$

from equilibrium in the x-direction,

$$\int \left[(\lambda + 2G)\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}\, w_{zJ} + a^2 G N_I N_J\, w_{zJ} + ia\lambda\frac{\partial N_I}{\partial z}N_J w_{xJ} - iaGN_I\frac{\partial N_J}{\partial z}w_{xJ} - B\frac{\partial N_I}{\partial z}N_J\, u_{wJ}\right]dz = S_{zI} \tag{23}$$

from equilibrium in the z-direction, and

$$\int \left[\frac{k_{zz}}{\gamma_w}\frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z} - ia\frac{k_{xz}}{\gamma_w}N_I\frac{\partial N_J}{\partial z} + ia\frac{k_{zx}}{\gamma_w}\frac{\partial N_I}{\partial z}N_J + a^2\frac{k_{xx}}{\gamma_w}N_I N_J - i\omega C\, N_I N_J\right]u_{wJ}dz + \int \left[a\omega B N_I N_J w_{xJ} - i\omega B N_I\frac{\partial N_J}{\partial z}w_{zJ}\right]dz = Q_{wI} \tag{24}$$

from continuity principles incorporating Darcy's Law.

Substituting equations 16, 17, 18 into the above three mixed real & imaginary differential equations, then separating the real & imaginary terms, results in six differential equations, each with nodal harmonic dependent variables as follows:

$$\left(a^2(\lambda+2G)\bar{\Gamma}_{IJ} + G\bar{A}^{zz}_{IJ}\right)\bar{w}_{xcJ} - \left(aG\bar{B}^{zN}_{IJ} - a\lambda\bar{B}^{Nx}_{IJ}\right)\bar{w}_{zsJ} - aB\bar{\Gamma}_{IJ}\,\bar{u}_{wsJ} = \bar{S}_{xcI} \qquad (25)$$

$$\left(a^2(\lambda+2G)\bar{\Gamma}_{IJ} + G\bar{A}^{zz}_{IJ}\right)\bar{w}_{xsJ} + \left(aG\bar{B}^{zN}_{IJ} - a\lambda\bar{B}^{Nx}_{IJ}\right)\bar{w}_{zcJ} + aB\bar{\Gamma}_{IJ}\,\bar{u}_{wcJ} = \bar{S}_{xsI} \qquad (26)$$

$$\left((\lambda+2G)\bar{A}^{zz}_{IJ} + a^2G\bar{\Gamma}_{IJ}\right)\bar{w}_{zcJ} - \left(a\lambda\bar{B}^{zN}_{IJ} - aG\bar{B}^{Nx}_{IJ}\right)\bar{w}_{xsJ} - B\bar{B}^{zN}_{IJ}\,\bar{u}_{wcJ} = \bar{S}_{zcI} \qquad (27)$$

$$\left((\lambda+2G)\bar{A}^{zz}_{IJ} + a^2G\bar{\Gamma}_{IJ}\right)\bar{w}_{zsJ} + \left(a\lambda\bar{B}^{zN}_{IJ} - aG\bar{B}^{Nx}_{IJ}\right)\bar{w}_{xcJ} + B\bar{B}^{zN}_{IJ}\,\bar{u}_{wsJ} = \bar{S}_{zsI} \qquad (28)$$

$$\left(a^2\frac{k_{xx}}{\gamma_w}\bar{\Gamma}_{IJ} + \frac{k_{zz}}{\gamma_w}\bar{A}^{zz}_{IJ}\right)\bar{u}_{wcJ} - \left(a\frac{k_{zx}}{\gamma_w}\bar{B}^{zN}_{IJ} - a\frac{k_{xz}}{\gamma_w}\bar{B}^{Nz}_{IJ} - \omega C\bar{\Gamma}_{IJ}\right)\bar{u}_{wsJ} + a\omega B\bar{\Gamma}_{IJ}\,\bar{w}_{xcJ} + \omega B\bar{B}^{Nz}_{IJ}\,\bar{w}_{zsJ} = \bar{Q}_{wcI} \qquad (29)$$

$$\left(a^2\frac{k_{xx}}{\gamma_w}\bar{\Gamma}_{IJ} + \frac{k_{zz}}{\gamma_w}\bar{A}^{zz}_{IJ}\right)\bar{u}_{wsJ} + \left(a\frac{k_{zx}}{\gamma_w}\bar{B}^{zN}_{IJ} - a\frac{k_{xz}}{\gamma_w}\bar{B}^{Nz}_{IJ} - \omega C\bar{\Gamma}_{IJ}\right)\bar{u}_{wcJ} + a\omega B\bar{\Gamma}_{IJ}\,\bar{w}_{xsJ} - \omega B\bar{B}^{Nz}_{IJ}\,\bar{w}_{zcJ} = \bar{Q}_{wsI} \qquad (30)$$

where:

$$\bar{A}^{zz}_{IJ} = \int \frac{\partial N_I}{\partial z}\frac{\partial N_J}{\partial z}\,dz \qquad (31)$$

$$\bar{\Gamma}_{IJ} = \int N_I\,N_J\,dz \qquad (32)$$

$$\bar{B}^{xN}_{IJ} = \int \frac{\partial N_I}{\partial x}N_J\,dz \qquad (33)$$

$$\bar{B}^{Nx}_{IJ} = \int N_I\frac{\partial N_J}{\partial x}\,dz \qquad (34)$$

5 INCORPORATION INTO A FINITE ELEMENT MODEL

A two-noded linear finite element shape function is sufficient to approximate the above matrix equations (Thomas, 1989, 1995). This simulates accurately the displacement, stress and pore water pressure, both in-phase and out-of-phase of the harmonic loading wave.

The above Galerkin Finite Element formulation has been programmed into the software WISPP (Wave Induced Stresses & Pore Pressures, Thomas, 1988).

For poro-elastic soil properties, applied boundary conditions loading, wavelength and angular velocity, WISPP produces a graphical output of the in-phase and out-of-phase x & z-displacements and pore-pressure at each of the nodes over the seabed depth, and the x-x, z-z, x-z operative stresses at each element centroid.

6 EXAMPLE OF WAVE LOADING ON A TWO-LAYERED SOIL

Figure 2 depicts finite element modelling with linear elements, of a multi-layered saturated soil with a 5m upper layer of drained bulk modulus K_1 = 1 MPa, shear modulus G_1 = 750 kPa, overlying a second 10m layer with higher stiffness of K_2 = 4 MPa & G_2 = 3 MPa.

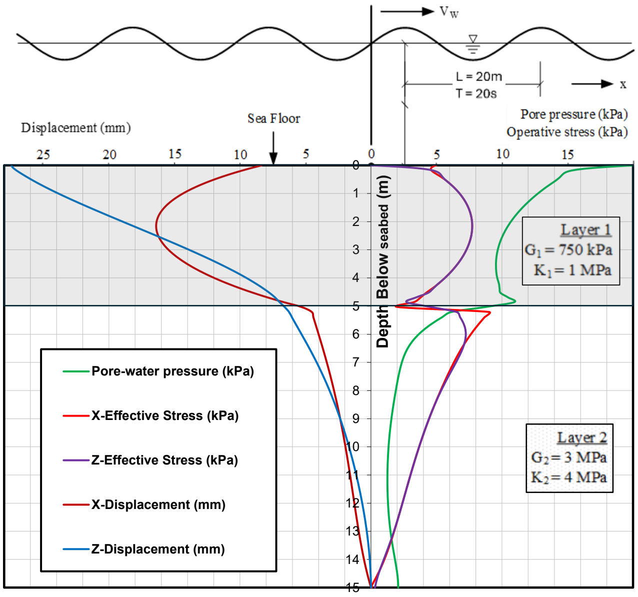

Figure. 2. Wave-induced x & z displacements, x & z operative stresses and pore-water pressures in a two-layer soil, simulated using the one-dimensional harmonic finite element WISSP model (Thomas, 1988).

A harmonic wave load of amplitude 20 kPa is applied to the soil surface with wavelength 20 m and wave period 20 s. The vertical domain has been discretised into 300 finite elements, with 100 elements in the upper layer and 200 elements in the lower layer. The permeability coefficient is 1 x 10^{-5} m/s for the upper layer, and 1 x 10^{-4} m/s for the lower layer. The output demonstrates that heterogeneity resulting from two layers with differing elastic moduli and permeability results in harmonic changes in effective stress, pore-pressure and soil displacement that are quite unexpected unless a harmonic model is applied.

7 CONCLUSIONS

Wave loading on a multi-layered poro-elastic seabed can result in unexpected displacements, stresses and pore-pressure behaviour. This is especially the case for a gassy soil seabed where undrained soil compression occurs. A Finite Element Model has been developed to simulate such behaviour under harmonic wave loading conditions using complex number theory.

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